NECESSARY AND SUFFICIENT CONDITIONS FOR CARLSON'S THEOREM ON ENTIRE FUNCTIONS

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1. *Introduction.*—It is our main purpose to outline a proof of the following theorem, for which only the necessity of condition (1.4) has previously been proved. It has long been suspected that (1.4) is not sufficient.

THEOREM 1. *In order that each entire function $f(z)$ satisfying the conditions*

$$f(z) = 0(1)e^{\tau|z|} \quad \textit{for some } \tau < \infty, \tag{1.1}$$

$$f(iy) = 0(1)e^{c|y|} \quad \textit{for some } c < \pi, \tag{1.2}$$

$$f(n) = 0 \quad \textit{for each } n \textit{ in } A, \tag{1.3}$$

vanish identically, it is necessary and sufficient that

$$\bar{D}(A) = 1, \tag{1.4}$$

where $\bar{D}(A)$ is the upper density of the set A.

The upper density $\bar{D}(A)$ is defined as usual by the formula

$$\bar{D}(A) = \limsup_{t \to \infty} \frac{A(t)}{t}, \tag{1.5}$$

where $A(t)$ is the number of integers n in A such that $0 < n \leq t$. Theorem 1 provides an optimal extension of a fundamental theorem of Carlson which states that if $f(z)$ is an entire function satisfying (1.1) and (1.2) and if $f(n) = 0$ for each positive integer n, then $f(z) \equiv 0$. W. H. J. Fuchs[1] has obtained a necessary and sufficient condition that each function $f(z)$ regular in the right half-plane and satisfying (1.1), (1.2), and (1.3) vanish identically. His condition is different from ours, so that there exists a set A of positive integers satisfying (1.4) but not Fuchs's condition.

Our proof of Theorem 1 will depend upon $\bar{D}_L(A)$, a modified logarithmic density of A, which is defined by

$$\bar{D}_L(A) = \inf_{\lambda > 1} \limsup_{x \to \infty} \frac{1}{\log \lambda} \sum_{n=x}^{\lambda x}{}^{*} \frac{1}{n}. \tag{1.6}$$

In (1.6) and formulas which follow, a star on a $\sum$ indicates that the index of summation is restricted to lie in A. We prove

THEOREM 2. *Theorem 1 remains valid if we replace* (1.4) *by the condition* $\bar{D}_L(A) =$ 1.

THEOREM 3. *For any set A of positive integers, $\bar{D}(A) = 1$ if and only if $\bar{D}_L(A) = 1$.*

Theorem 1 then follows from Theorems 2 and 3.

2. *Proof of the Sufficiency of the Condition $\bar{D}_L(A) = 1$.*—We now assume that $\bar{D}_L(A) = 1$ and prove that each entire function satisfying (1.1), (1.2), and (1.3) must vanish identically. Suppose, on the contrary, that there exists an entire function $f(z)$ satisfying (1.1), (1.2), and (1.3) for which $f(z) \not\equiv 0$. There is no loss of generality in assuming that $f(0) = 1$. Let the zeros of $f(z)$ be denoted by z_1,

$z_2, \ldots$, where $z_n = r_n \exp(i\theta_n)$. A modification of Carleman's theorem yields the inequality

$$\sum_{n=t}^{\lambda t}{}^* \frac{1}{n} \leq (\lambda t)^{-2} \sum_{r_n \leq \lambda t} r_n + \frac{1}{\pi \lambda t} \int_{-\pi/2}^{\pi/2} \log |f(\lambda t e^{i\theta})| \cos\theta\, d\theta - \frac{1}{\pi t} \int_{-\pi/2}^{\pi/2} \log |f(te^{i\theta})| \cos\theta\, d\theta + \frac{1}{2\pi} \int_t^{\lambda t} \{y^{-2} - (\lambda t)^{-2}\} \log |f(iy)f(-iy)|\, dy + \frac{1}{2\pi} \int_0^t \{t^{-2} - (\lambda t)^{-2}\} \log |f(iy)f(-iy)|\, dy = \textstyle\sum_1 + \int_1 - \int_2 + \int_3 + \int_4. \quad (2.1)$$

Known techniques[2] yield the estimates $\sum_1 \leq K$, $\int_1 \leq K$, $\int_2 \geq -K$, $\int_3 \leq (c/\pi) \log \lambda + K$, $\int_4 \leq K$. Here $K = K(c, \tau, t)$ is independent of λ and is bounded for large t. Applying these estimates to (2.1) and letting first t and then λ approach infinity, we see that $\bar{D}_L(A) \leq (c/\pi) < 1$, contradicting our hypothesis.

3. *Proof of the Necessity of the Condition $\bar{D}_L(A) = 1$.*—We now assume that each entire function $f(z)$ satisfying (1.1), (1.2), and (1.3) vanishes identically, and prove that $\bar{D}_L(A) = 1$. Suppose, on the contrary, that $\bar{D}_L(A) < 1$. We prove the following lemma.

LEMMA 1. *If $\bar{D}_L(A) < 1$, then $\bar{D}(A) < 1$.*

To prove this lemma, we choose a number ψ for which $\bar{D}_L(A) < \psi < 1$. We may choose a number $\lambda > 1$ and then a number $M > 0$, so that

$$\sum_{n=t}^{\lambda t}{}^* \frac{1}{n} < \psi \log \lambda \qquad \text{for } t > M. \quad (3.1)$$

We shall prove that

$$\bar{D}(A) \leq \frac{\lambda - \lambda^{1-\psi}}{\lambda - 1} < 1. \quad (3.2)$$

Let $t > M$, and put $t = \alpha\lambda^p$, where $M \leq \alpha < \lambda M$ and p is a nonnegative integer. Put $B_k = A(\alpha\lambda^{k+1}) - A(\alpha\lambda^k)$, $k = 0, 1, \ldots, p-1$, so that $A(t) \leq \alpha + \sum_{k=0}^{p-1} B_k$. Using (3.1) to estimate the B_k, we find that

$$\psi \log \lambda \geq \sum_{\alpha\lambda^k}^{\alpha\lambda^{k+1}}{}^* \frac{1}{n} \geq \sum_{\alpha\lambda^{k+1} - B_k}^{\alpha\lambda^{k+1}} \frac{1}{n} \geq \log \frac{\alpha\lambda^{k+1}}{\alpha\lambda^{k+1} - B_k + 2},$$

which implies that $B_k \leq 2 + \alpha\lambda^{k+1}\{1 - \lambda^{-\psi}\}$, $A(t) = A(\alpha\lambda^p) \leq \alpha\{1 - \lambda^{-\psi}\} \lambda(\lambda^p - 1)/(\lambda - 1) + \alpha + 2p$, and hence that (3.2) holds. This completes the proof of the lemma.

The conclusion $\bar{D}(A) < 1$ implies[3] that the function $f_A(z)$, defined by

$$f_A(z) = \prod_{n \in A} \left(1 - \frac{z^2}{n^2}\right),$$

satisfies (1.1), (1.2), and (1.3) and therefore violates our hypothesis, establishing the necessity of the condition $\bar{D}_L(A) = 1$.

4. *Proof of Theorems 1 and 3.*—We can now complete the proofs of Theorems 1 and 3 by proving Theorem 3. First, suppose that $\bar{D}(A) = 1$. Then Lemma 1 implies that $\bar{D}_L(A) = 1$. Suppose, next, that $\bar{D}_L(A) = 1$. Then each entire func-

tion satisfying (1.1), (1.2), and (1.3) must vanish identically. But if $\bar{D}(A) < 1$, then the function $f_A(z)$ above satisfies (1.1), (1.2), and (1.3), and the condition $f_A(0) = 1$. Hence $\bar{D}(A) = 1$. This completes the proofs of the theorems.

[1] *J. London Math. Soc.*, **21**, 106–110, 1946.

[2] The estimates for $\int_1$, $\int_3$, and $\int_4$ follow easily from (1.1) and (1.2). The key steps in estimating Σ_1 and $\int_2$ may be found in R. P. Boas, Jr., *Entire Functions* (New York, 1954), pp. 16, 31.

[3] R. C. Buck, *Duke Math. J.*, **13**, 345–349, 1946.